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Magnetic susceptibility of d.c. SQUID's

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Abstract. We study the magnetic susceptibility of a conventional d.c. SQUID adopting a description based on the reduced two-junction interferometer model. In this context, considering the parameter β as a perturbation parameter, it is possible to derive an analytical expression for magnetic susceptibility with respect to the externally applied magnetic flux. Numerical simulations on the complete model has been carried out in order to confirm the analytical results.

PACS. 74.50.+r Tunneling phenomena; point contacts, weak links, Josephson effects – 85.25.Dq Superconducting quantum interference devices (SQUIDs)

1 Introduction

Superconducting Quantum Interference Devices (SQUID's) [1] are very interesting dynamical systems "per se". The simplest description of the magnetic response of a conventional d.c. SQUID relies upon the two-junction interferometer model with $\beta = 0$. According to this model the time evolution of the average superconducting phase φ_A of the two junctions in the device can be expressed by the following non-linear differential equation in the framework of the RSJ model [2]:

$$\frac{d\varphi_A}{d\tau} + (-1)^n \cos(\pi\Psi_{ex}) \sin(\varphi_A) = \frac{i_B}{2}, \qquad (1)$$

where τ is the normalized time variable, n is an integer, Ψ_{ex} is the applied magnetic flux normalized to the elementary flux quantum Φ_0, and i_B is the bias current normalized to the maximum Josephson current I_J of both junctions. When discussing the dynamical behaviour of d.c. SQUID's containing one π-junction [3] the above model is modified by making the following substitution [4]:

$$\Psi_{ex} \to \Psi_{ex} + \frac{2k+1}{2}, \qquad (2)$$

where k is an integer. Equation (1) is formally identical to the dynamical equation of a single overdamped junction having maximum Josephson current equal to $I_J |\cos(\pi\Psi_{ex})|$ (conventional d.c. SQUID) or to $I_J |\sin(\pi\Psi_{ex})|$ (π-SQUID). The present analysis can thus be extended to π-SQUID's.

Even though most of the electrodynamic properties of d.c. SQUID's can be captured by equation (1), a recently developed more detailed model [5] takes explicitly account the effects due to finite β values (we here set $\beta = \frac{LI_J}{\Phi_0}$, L being the inductance of a single loop branch). This model goes under the name of reduced two-junction interferometer model and considers, to first order in the parameter β, the time evolution of both φ_A and the normalized flux Ψ linked to the SQUID as follows:

$$\frac{d\varphi_A}{d\tau} + (-1)^n \cos(\pi\Psi_{ex}) \sin(\varphi_A) + \pi\beta \sin^2(\pi\Psi_{ex}) \sin(2\varphi_A) = \frac{i_B}{2}, \qquad (3a)$$

$$\Psi = \Psi_{ex} - 2(-1)^n \beta \sin(\pi\Psi_{ex}) \cos(\varphi_A). \qquad (3b)$$

In equation (3a) we notice the appearance of an additional second harmonic term as a consequence of a more accurate determination of the flux Ψ in equation (3b). Naturally, equations (3a, 3b) reproduce the $\beta = 0$ two-junction interferometer model for null values of the parameter β. Finally, by simply making the substitution for Ψ_{ex} given in equation (2), equations (3a, 3b) reduce to an analogous set of equations for π SQUID's [6]. In the present work we start by noticing that adoption of the reduced two-junction interferometer model makes it possible to analytically calculate the expression of the susceptibility of conventional d.c. SQUID to first order in the parameter β. Numerical integration of the complete model (without any approximation) is performed to confirm the analytic procedure.

2 Magnetic susceptibility of conventional d.c. SQUID's

In the reduced two-junction interferometer model, the dynamics of the average value of the superconducting phase

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difference φ_A across the two branches of a conventional d.c. SQUID is governed by a first-order non-linear differential equation (Eq. (3)), while the normalized total magnetic flux Ψ is fully determined once we know the expression of φ_A according to equation (3b). It is thus possible to derive the analytic expression of the d.c. magnetic susceptibility χ in the following form:

$$\chi = \frac{\Psi - \Psi_{ex}}{\Psi_{ex}} = -2\pi\beta \frac{\sin(\pi\Psi_{ex})}{\pi\Psi_{ex}} \cos\tilde{\varphi}_A, \quad (4)$$

where $\tilde{\varphi}_A$ represents the average phase difference across the electrodes in the steady state. We can observe that for $\beta = 0$ we have a magnetic susceptibility exactly equal to zero, while for small but finite values of β we are able to study the dependence of χ on the externally applied flux. We start by noticing that in the running state (when a voltage different from zero is detected across the device), $\tilde{\varphi}_A$ is a function of time and so does the magnetic susceptibility. We can evaluate the time average of χ, which is found to be a quantity of higher order than one in β. Besides this trivial case, we are interested in the description of the system magnetic response in the zero voltage-state (a null voltage is observed across the system, while a non-zero current is present). For this state we could distinguish two cases: Case a) $i_B = 0$; case b) $i_B \neq 0$.

Case a)

In the zero-voltage state and for $i_B = 0$, $\tilde{\varphi}_A$ is the stable solution of the stationary portion of equation (3a):

$$(-1)^n \cos(\pi\Psi_{ex}) \sin(\varphi_A) + \pi\beta \sin^2(\pi\Psi_{ex}) \sin(2\varphi_A) = 0, \quad (5)$$

to be solved in the interval $[0, 2\pi[$ with respect to the variable φ_A. The above equation has the following solutions: $\tilde{\varphi}_A = 0$ and $\tilde{\varphi}_A = \pi$, if $\frac{|\cos(\pi\Psi_{ex})|}{2\pi\beta\sin^2(\pi\Psi_{ex})} > 1$; $\tilde{\varphi}_A = 0$, $\tilde{\varphi}_A = \pi$, $\cos^{-1}\left(\frac{(-1)^n\cos(\pi\Psi_{ex})}{2\pi\beta\sin^2(\pi\Psi_{ex})}\right)$ and $2\pi - \cos^{-1}\left(\frac{(-1)^n\cos(\pi\Psi_{ex})}{2\pi\beta\sin^2(\pi\Psi_{ex})}\right)$, if $\frac{|\cos(\pi\Psi_{ex})|}{2\pi\beta\sin^2(\pi\Psi_{ex})} < 1$. Therefore, we must adopt a rule which allows us to choose between these solutions.

Let us first study the case with n even. By analysing the sign of $\dot{\varphi}_A$ when $\frac{|\cos(\pi\Psi_{ex})|}{2\pi\beta\sin^2(\pi\Psi_{ex})} > 1$, we can notice that, for $\cos(\pi\Psi_{ex}) > 0$, the steady state is reached for $\tilde{\varphi}_A = 0$, $\tilde{\varphi}_A = \pi$ being an unstable solution. For $\cos(\pi\Psi_{ex}) < 0$, on the other hand, the stationary stable solution is obtained for $\tilde{\varphi}_A = \pi$, while $\tilde{\varphi}_A = 0$ is unstable. Summarising, we can write the analytical expression of χ as follows:

$$\chi = \begin{cases} -2\pi\beta\frac{\sin(\pi\Psi_{ex})}{\pi\Psi_{ex}} \text{ if } \cos(\pi\Psi_{ex}) > 0 \\ 2\pi\beta\frac{\sin(\pi\Psi_{ex})}{\pi\Psi_{ex}} \text{ if } \cos(\pi\Psi_{ex}) < 0 \end{cases}, \quad (6)$$

while for $\cos(\pi\Psi_{ex}) = 0$ both values of the susceptibility χ given in equation (6) are possible.

By analysing now the sign of $\dot{\varphi}_A$ when $\frac{|\cos(\pi\Psi_{ex})|}{2\pi\beta\sin^2(\pi\Psi_{ex})} < 1$, we find that both states corresponding to $\tilde{\varphi}_A = 0$ and $\tilde{\varphi}_A = \pi$ are stable. In these regions, centred at half-integer values of the applied flux and with amplitude about equal to 4β, the magnetic states are bistable and thus the magnetic response of the device depends upon the system history. In order to show all possible states of the system, following Strogatz [7], in Figure 1 we represent the stationary solutions as a function of the normalized applied flux.

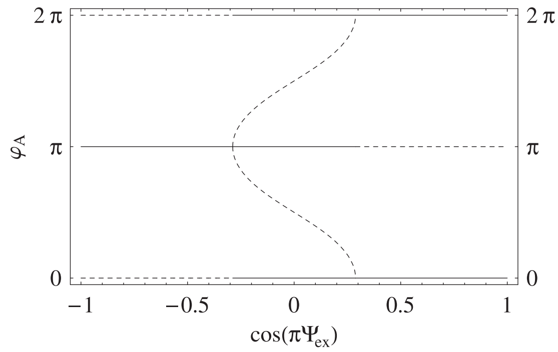

Fig. 1. Fixed points diagram for the system described by equation (3a) with n even, $\beta = 0.05$ and $i_B = 0$. Stable solutions are described by full lines, while unstable ones are described by dashed lines. Notice appearance of bistability in the interval $-\left(\sqrt{\frac{1}{16\pi^2\beta^2}+1} - \frac{1}{4\pi\beta}\right) < \cos(\pi\Psi_{ex}) < \left(\sqrt{\frac{1}{16\pi^2\beta^2}+1} - \frac{1}{4\pi\beta}\right)$.

For n odd we can adopt a similar analysis. By doing so, we notice that the system behaves as in the n even case including field values for which $\frac{|\cos(\pi\Psi_{ex})|}{2\pi\beta\sin^2(\pi\Psi_{ex})} < 1$ where, once again, bistability is present.

Case b)

Considering equation (3a), we can notice that, for $\cos(\pi\Psi_{ex}) = 0$ and for $|i_B| > 2\pi\beta$, the variable φ_A varies with respect to time, so that magnetic susceptibility becomes an oscillating quantity. It can be shown, by means of direct calculation, that the time average of χ is zero. If we now consider the case in which $\cos(\pi\Psi_{ex}) \neq 0$, we can solve perturbatively the stationary portion of equation (3a) and write the following relation:

$$\sin(\tilde{\varphi}_A) = \frac{i_B}{2\cos(\pi\Psi_{ex})} + o(\beta), \quad (7)$$

where only the zero-th order term is shown, given that the susceptibility already contains a factor β in equation (4). Proceeding as before, we can write, to first order in β, the expression for d.c. magnetic susceptibility in this case as follows:

$$\chi = \begin{cases} -2\pi\beta\frac{\sin(\pi\Psi_{ex})}{\pi\Psi_{ex}}\sqrt{1-\left(\frac{i_B}{2\cos(\pi\Psi_{ex})}\right)^2} \text{ if } \cos(\pi\Psi_{ex}) > 0 \\ 2\pi\beta\frac{\sin(\pi\Psi_{ex})}{\pi\Psi_{ex}}\sqrt{1-\left(\frac{i_B}{2\cos(\pi\Psi_{ex})}\right)^2} \text{ if } \cos(\pi\Psi_{ex}) < 0 \end{cases}. \quad (8)$$

It's worth noting that equation (8) defines a real susceptibility function only for $1 - \left(\frac{i_B}{2\cos(\pi\Psi_{ex})}\right)^2 \geq 0$, i.e., only when the system is in the zero-voltage state. When this condition is broken, magnetic susceptibility becomes an oscillating null-average function. Notice also that, for $i_B = 0$, equation (8) reduces to equation (6). When $\cos(\pi\Psi_{ex}) = 0$, considering the stationary portion of equation (3a), we obtain, for $|i_B| < 2\pi\beta$, the following four solutions:

$$\cos(\tilde{\varphi}_A) = \pm\sqrt{\frac{1 \pm \sqrt{1 - \left(\frac{i_B}{2\pi\beta}\right)^2}}{2}}, \tag{9}$$

of which two are stable and two unstable. The two stable solutions correspond to two possible magnetic states of the system giving the following values for χ:

$$\chi = \pm\frac{4\beta}{2k+1}\sqrt{\frac{1 + \sqrt{1 - \left(\frac{i_B}{2\pi\beta}\right)^2}}{2}}, \tag{10}$$

where k is an integer fixed by the normalized external applied flux value ($\Psi_{ex} = \frac{2k+1}{2}$). In the $i_B \neq 0$ case, however, these susceptibility values are isolated points in the χ vs. Ψ_{ex} curves.

As before, if we consider the case for n odd, we notice a complete identical behaviour of the system up to first order in the parameter β.

3 Numerical results

In order to make a comparison between the above analytical results and numerical results coming from the complete SQUID model, we have simulated the complete dynamics of a conventional d.c. SQUID within the RSJ model. In particular, we have computed the time average of the magnetic susceptibility χ as a function of applied magnetic flux by starting from zero-field cooled (ZFC) conditions ($\Psi = 0$ at $\Psi_{ex} = 0$) and by gradually increasing (decreasing) Ψ_{ex} up to a maximum (minimum) value. In Figure 2a and 2b we report χ vs. Ψ_{ex} curves for $\beta = 0.01$ and respectively for $i_B = 0$ and $i_B = 0.8$. We notice that the system shows the analytic behaviour reported in equations (6) and (8), when it is in the zero-voltage state, while it presents a null value of the susceptibility $\left(o\left(\beta^2\right)\right)$ when it is in the running state. In order to get some information on the limits of validity of the model, we run a second simulation for $\beta = 0.05$, obtaining the χ vs. Ψ_{ex} curves for $i_B = 0$ and $i_B = 0.8$ of Figures 3a, and 3b, respectively. As can be seen from Figures 3a, 3b, even though the analytical solution fails for $|\Psi_{ex}| < \frac{1}{2}$, a good agreement between the numerical results (obtained for the complete system, without any approximation) and the analytic results of equations (6) and (8), is attained only for flux values $|\Psi_{ex}| > \frac{1}{2}$.

Apart from the limits of validity of the analytic approach, Figures 2a, 2b and 3a, 3b show that the magnetic field susceptibility of conventional d.c. SQUID's presents alternating signs, with the appearance of a paramagnetic response for well-defined applied flux intervals. This should not be considered a surprising feature, given that we are dealing with a weakly-coupled multiply-connected superconducting structure. Indeed, it can be shown that even type I superconductors, presenting a multiply connected topology, may show paramagnetic response for some interval of the applied magnetic flux [8]. We now turn our attention to the bistability of the stationary solutions for the average phase difference φ_A found in the previous section. This feature does not appear in Figures 2a, 2b and 3a, 3b, since we have chosen to start from a ZFC state of the system and have raised (lowered) the field up to its maximum (minimum) value only in one direction. Therefore, in order to make this feature explicitly evident, in Figures 4a, 4b we show the magnetic behaviour of the system in the bistability region close to $\Psi_{ex} = \frac{3}{2}$, for $\beta = 0.01$ and $i_B = 0$. In Figures 4a and 4b, obtained for $n = 0$ and $n = 1$, respectively, the magnetic susceptibility attains the same values, following the same hysteretic paths. This confirms the similarity of the magnetic behaviour of this system at different values of n.

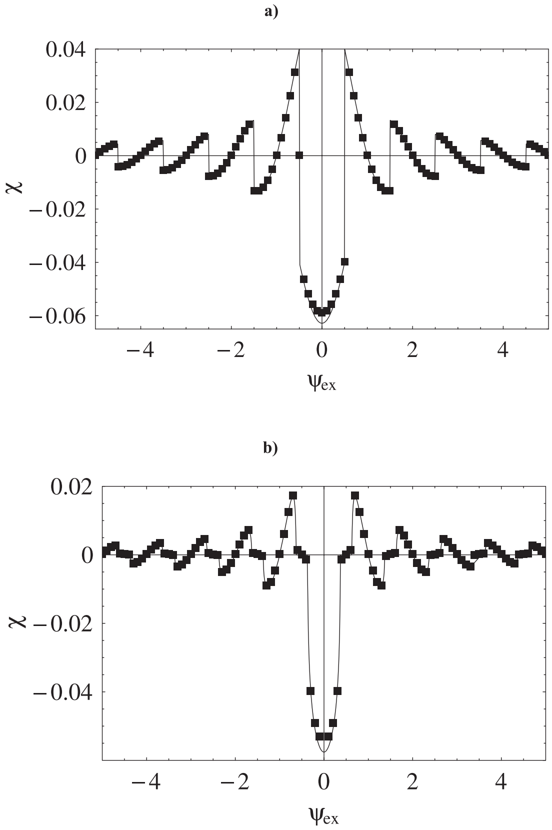

Fig. 2. (a) χ vs. Ψ_{ex} curves for $i_B = 0$ and $\beta = 0.01$. The full line represents the analytical solution given in equation (6) in the text, while the dots are obtained by means of numerical integration of the complete dynamical equations of a conventional d.c. SQUID. (b) χ vs. Ψ_{ex} curves for $i_B = 0.8$ and $\beta = 0.01$. The full line represents the analytical solution given in equation (8) in the text, while the dots are obtained by means of numerical integration of the complete dynamical equations of a conventional d.c. SQUID.

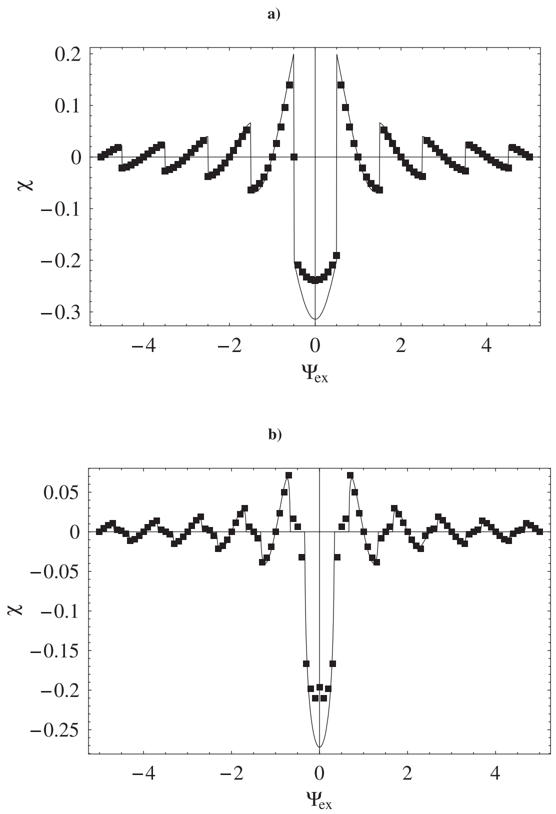

Fig. 3. (a) χ vs. Ψ_{ex} curves for $i_B = 0$ and $\beta = 0.05$. The full line represents the analytical solution given in equation (6) in the text, while the dots are obtained by means of numerical integration of the complete dynamical equations of a conventional d.c. SQUID. (b) χ vs. Ψ_{ex} curves for $i_B = 0.8$ and $\beta = 0.05$. The full line represents the analytical solution given in equation (8) in the text, while the dots are obtained by means of numerical integration of the complete dynamical equations of a conventional d.c. SQUID.

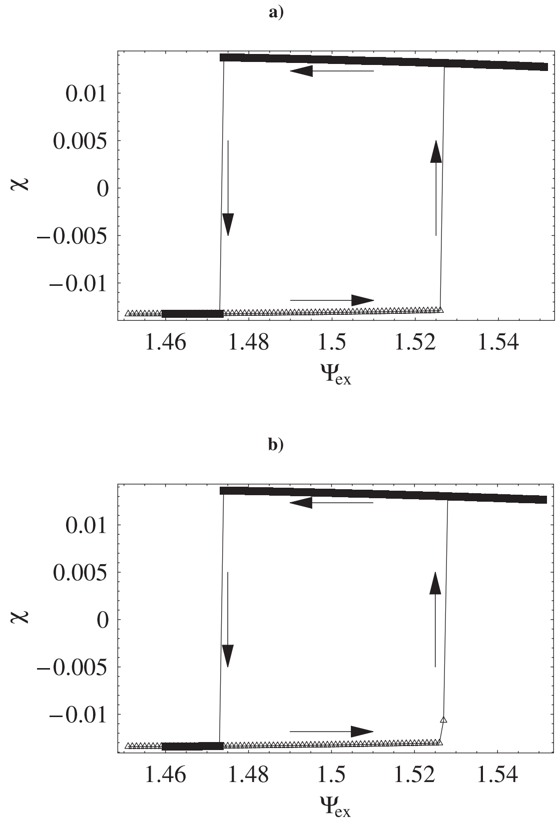

Fig. 4. (a) χ vs. Ψ_{ex} curves for $n = 0$, $i_B = 0$ and $\beta = 0.01$ in the vicinity of $\Psi_{ex} = \frac{3}{2}$. Triangles represent magnetic states for increasing field, while black boxes represent magnetic states for decreasing field. (b) χ vs. Ψ_{ex} curves for $n = 1$, $i_B = 0$ and $\beta = 0.01$ in the vicinity of $\Psi_{ex} = \frac{3}{2}$. Triangles represent magnetic states for increasing field, while black boxes represent magnetic states for decreasing field.

4 Conclusions

We studied, by means of the reduced two-junction interferometer model, the magnetic susceptibility χ of conventional d.c. SQUID's. Under the assumption that the parameter β is not identically equal to zero, we find an approximated analytic expression for χ to first order in β. For a conventional d.c. SQUID there exists a very low field region in which the magnetic response of the system is always diamagnetic. The extension $\Delta\Psi_{ex}$ of the Ψ_{ex} interval centred at zero in which the response is diamagnetic depends on $i_B (i_B < 2)$ as follows:

$$\Delta\Psi_{ex} = \frac{2}{\pi}\cos^{-1}\left(\frac{i_B}{2}\right).$$

Outside this region there exists an alternating behaviour between paramagnetic and diamagnetic responses. On the other hand, the system shows hysteretic behaviour in intervals of Ψ_{ex} centred on semi-integer values with amplitude of the order of β.

The richness of the magnetic response of this system appears only when the $\beta = 0$ hypothesis is removed. The analysis carried out, however, is only valid to first order in β, so that further studies need to be done in order to study the magnetic behaviour of the system at arbitrary values of this parameter.

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